VaR and Ruin Probability

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Introduction–VaR:

Consider an insurance risk setting:

- Let $\{S(t)\}_{t\geq 0}$ denote the aggregate operating losses of a company during time $(0, t]$.
- In VaR literature, one usually concerns the random variable $S(t)$ for fixed time t. The value at risk at level α is defined as $VaR_{\alpha}(S(t)) = \inf\{l, \mathbb{P}(S(t) > l) < 1-\alpha\} = \inf\{l, F_{S(t)}(l) \geq \alpha\}$.
- The probability level α may assume values 0.90, 0.95, or 0.99 etc. For example, if a company's capital level is $VaR_{0.9}(S(1))$, then there is 90% chance the company will be able to cover its possible operating losses next time period.

VaR

- VaR is intended to be a risk measure of financial distress over a short period of time. (Pan and Duffie, 1997)
  - In finance, the time horizon is usually a number of days. For example, the Bank for International Settlements (BIS) set p to 99% and t to ten days for purposes of measuring the adequacy of bank capital. Many firms use an overnight VaR for internal purposes.
  - In insurance, Solvency II requires a 99.5% **one year** VaR.
  - Notice that the time horizon in the insurance setting is much larger than that used in a bank setting, perhaps because insurance transactions are much less frequent than banking transactions.

Criticism of VaR

- VaR ignores what happens in the tails. It specifically cuts them off. A 99% VaR calculation does not evaluate what happens in the last 1%. (Einhorn 2008)
- By ignoring the tails, VaR creates an incentive to take excessive but remote risks. (Einhorn 2008)

Criticism of VaR–Example

- Consider underwriting two potential (annual) losses X and Y, where X takes value 1000 with $p = 0.001$ and zero otherwise; Y takes value 10000 with $p = 0.0001$ and zero otherwise. An insurer can charge a premium 2 for risk X and 10 for risk Y.
- The annual aggregate operating loss random variables in the two situations are $S_1(1) = X - 2$ and $S_2(1) = Y - 10$.
- $VaR_{0.99}(S_1(1)) = -2$ and $VaR_{0.99}(S_2(1)) = -10$. That is, you don't need any capital to support underwriting the risk.
- A firm has the incentive to take risk Y for extra profit if capital requirement is determined by VaR – remote risk is ignored by VaR.

Criticism of VaR–Example

- A remedy for this is the use of TVaR, defined by $TVaR_\alpha(S(t)) = \mathbb{E}(S(t)|S(t) > VaR_\alpha(S(t)))$.
- For our example,

$$TVaR_{0.99}(S_1(1)) = \mathbb{E}(S_1(1)|S_1(1) > VaR_{0.99}(S(t))) = 1000,$$

$$TVaR_{0.99}(S_2(1)) = 10,000.$$

- This means that Y is riskier than X according to TVaR.

Criticism of TVaR–Example

- Consider two risks, X and Y: X takes value 600 with $p = 0.001$ and zero otherwise; Y takes value 1000 with probability 0.0005, 200 with probability $p = 0.0005$ and zero otherwise.
- Suppose one may charge a premium of 2 for risk X and 5 for risk Y. Then the annual aggregate losses become $S_1(1) = X - 2$ and $S_2(1) = Y - 5$.
- $VaR_{0.99}(S_1(1)) = -2$ and $VaR_{0.99}(S_2(1)) = -5$.
- $TVaR_{0.99}(S_1(1)) = TVaR_{0.99}(S_2(1)) = 600$.
- This example shows that TVaR can also ignore tail risk.

Introduction–Ruin probability

- Next, we show that infinite time horizon ruin probability is naturally a remedy for this problem.
- Instead of judging how risky it is to underwrite the risk for one year, ruin theorists ask how risky it is to continue to run the same business indefinitely.

Binomial Risk Model

- Consider running the insurance company for t years. Assume that in each year, there is a claim with probability p or no claim with probability $q = 1 - p$. Assume that the annual premium is one.
- Then the aggregate operating losses at year t can be modeled by the so called compound binomial risk model (Gerber, 1988):

$$S(t) = (X_1 + \cdots + X_{N(t)}) - t,$$

  where $t = 1, 2, 3, \cdots$ and N_t is the number of claim in the first t periods.
- Ruin is the event that $S(t) \geq u$ for some $t \geq 1$, where u is the initial surplus.

Example 1

We consider two cases

- case (1): (denoted by $S_1(t)$), $p = 0.001$ and the claim sizes $X_i, i = 1, 2, \cdots$ be fixed value 600.
- case (2): (denoted by $S_2(t)$), $p = 0.002$ and the claim sizes $X_i, i = 1, 2, \cdots$ be fixed value 300.
- $VaR_{0.99}(S_1(1)) = VaR_{0.99}(S_2(1)) = -1.$
- Ruin probability $\psi_1(u) = \mathbb{P}(\sup_{t \geq 1} S_1(t) \geq u)$, where u is the initial surplus.

Example 1

- Gerber (1988) showed that $\psi_1(0) = p\mathbb{E}(X) = 0.6$ and $\psi_1(u) = q\psi_1(u+1) + p$, for $1 < u < 600$ and $\psi_1(u) = q\psi_1(u+1) + p\psi_1(u+1-600)$, for $u \geq 600$.
- $\psi_2(u)$ can be calculated similarly.
- Ruin probabilities as a function of initial surplus in plotted in figure 1.

Example 1

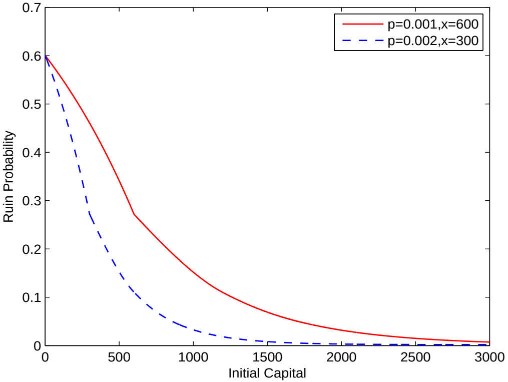

Figure: Ruin Probability as risk measure –example 1

- This figure shows that, when using ruin probability as the risk measure
  - $\{S_1(t)\}_{t\geq 0}$ is riskier than $\{S_2(t)\}_{t\geq 0}$
  - If one requires that the ultimate ruin probability to be less than certain level, say 0.1, then the required initial capital can be readily determined from the graph.

Example 2

We consider two cases

- case (1): (denoted by $S_1(t)$), $p = 0.001$ and the claim sizes $X_i, i = 1, 2, \cdots$ be fixed value 600.
- case (3): (denoted by $S_3(t)$), $p = 0.001$ and the claim sizes $X_i, i = 1, 2, \cdots$ take values 200 and 1000 with probability $1/2$.
- $VaR_{0.99}(S_1(1)) = VaR_{0.99}(S_3(1)) = -1$ and $TVaR_{0.99}(S_1(1)) = TVaR_{0.99}(S_3(1)) = 600$.

Example 2 continued

- In case (3), Gerber (1988) showed that
  $\psi_3(0) = p\mathbb{E}(X) = 0.6$ and
  $\psi_3(u) = q\psi_3(u+1) + p$, for $1 < u < 200$,
  $\psi_3(u) = q\psi_3(u+1) + p\psi_3(u+1-200)$, for $200 < u < 1200$
  and
  $\psi_3(u) = q\psi_3(u+1) + \frac{1}{2}p\psi_3(u+1-200) + \frac{1}{2}p\psi_3(u+1-1200)$, for $u \geq 600$.
- Ruin probabilities as a function of initial surplus in plotted in figure (2).

Example 2

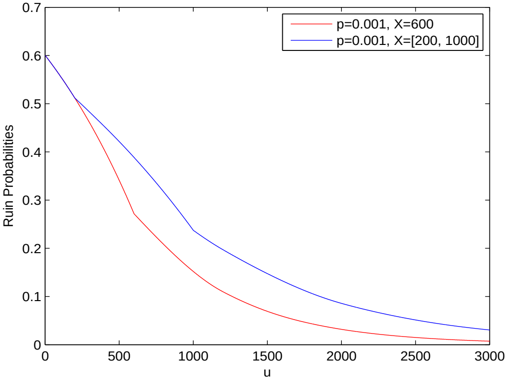

Figure: Ruin Probability as risk measure–example 2

VaR and Ruin probability as risk measures

- VaR is a risk measure of $S(t)$ for fixed t.
- In ruin theory literature, one usually concerns with the random variable $M(\infty) = \sup\{S(x), 0 \leq x\}$.
- the infinite time horizon ruin probability is defined by $\psi(u) = \mathbb{P}(M(\infty) > u)$. That is, ruin probability is a risk measure of $M(\infty)$
- These two measures provide different information about the risk in concern.
- Instead of judging how risky it is to bet on one trial of flipping a coin, ruin theorists ask how risky it is to continue betting on a lot of trials.

Brownian motion risk process

- Let $S(t) = -\mu t + \sigma W(t)$ be the aggregate operating losses, where $W(t)$ is a standard Brownian motion.
- $S(t) \sim N(-\mu t, \sigma^2 t)$.
- $VaR_p(S(t)) = -\mu t + \sigma t^{1/2} \Phi^{-1}(p)$.
- $TVaR_p(S(t)) = \mathbb{E}(S(t)|S(t) > VaR_p(S(t))) = -\mu t + \sigma t^{1/2} \frac{\phi(\Phi^{-1}(p))}{1-p}$.

Brownian motion risk process

- Infinite time horizon ruin probability concerns $M(\infty) = \sup_{t \geq 0}\{S(t)\}$.
- $F_{M(\infty)}(y) = 1 - e^{2\mu y/\sigma^2}$, for $\mu > 0$.
- We next illustrate how VaR and ruin probability differ in this case.

Example 3

- case 1 ($S_1(t)$): $\mu = -1$, $\sigma = 1$;
- case 2 ($S_2(t)$): $\mu = -10$, $\sigma = 4.8687$;
- $VaR_{0.99}S_1(1) = VaR_{0.99}S_2(1) = 1.3263$.
- Ruin probabilities as a function of initial surplus in plotted in figure (3).

Example 3

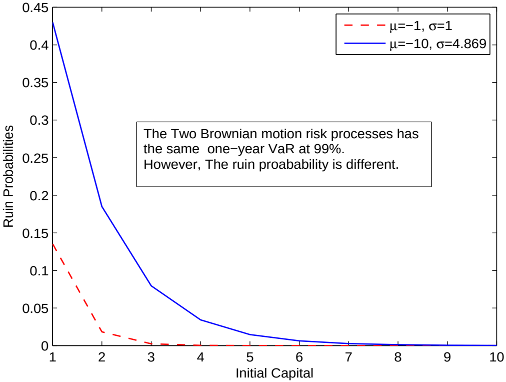

Figure: Ruin Probability as risk measure–example 4.

Conclusion of the examples

- VaR and TVaR consider the short term effect of a risk.
- By looking at the long term effect of the risk, ruin probability supplement VaR and TVaR as a informative risk measure.

VaR and Finite Time Ruin probability

- Define $M(t) = \sup\{S(x), 0 \leq x \leq t\}$. for some fixed t.
- The ruin probability with time horizon t is defined by $\psi(u, t) = \mathbb{P}(M(t) > u)$, where u is the insurer's initial capital level.
- The surplus level to ensure that the t year ruin probability is less than a small probability $1 - \alpha$ is $R_\alpha(S(t)) = \inf\{l, F_{M(t)}(l) \geq \alpha\} = VaR_\alpha(M(t))$.
- Obviously, $R_\alpha(S(t)) \geq VaR_\alpha(S(t))$

Analysis of the time horizon

- Is the one–year horizon used by Solvency II for insurance company too long?
- What is chance of something very bad occurs during $(0, t)$?

Analysis of the time horizon

- This question has been analyzed by Boukoudh et al. (2004), in which the authors argue that, with reasonable parameters, the interim risk ($M(t)$) could exceed $S(t)$ by 40%.

Analysis of the time horizon

- $M(t) = \sup\{S(x), 0 \leq x \leq t\}$.
- It is known that $F_{M(t)}(y) = \Phi\left(\frac{y+\mu t}{\sigma t^{1/2}}\right) - e^{-2\mu y/\sigma^2}\Phi\left(\frac{-y+\mu t}{\sigma t^{1/2}}\right)$. See for example, page 14 of Harrison (1985).
- Notice that $F_{S(t)}(y) = \Phi\left(\frac{y+\mu t}{\sigma t^{1/2}}\right)$.
- With this, we may compare $\psi(u, t) = \mathbb{P}(M(t) > u)$ with $\mathbb{P}(S(t) > u)$.

Analysis of the time horizon–an approximation

- For this simple case, the joint distribution of $S(t)$ and $M(t)$ is known, so that the relationship between $S(t)$ and $M(t)$ can be analyzed. however, we next consider a rough approximation.
- Instead of investigating the relationship between $S(t)$ and $M(t)$, we consider $S(\tau)$ and $M(\tau)$, where τ is an exponential random variable with mean t and is independent of $\{S(t), t \geq 0\}$.

Analysis of the time horizon–an approximation

- It is well–known that $M(\tau)$ and $M(\tau) - S(\tau)$ are independent and exponentially distributed with rates

$$\omega = \frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2}{\sigma^2 t}}$$

and

$$\eta = \frac{-\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2}{\sigma^2 t}}$$

respectively.

Analysis of the time horizon

- Proposition 1: $VaR_\alpha(M(\tau)) \sim -\frac{\log(1-\alpha)}{\frac{\mu}{\sigma^2}+\sqrt{\frac{\mu^2}{\sigma^4}+\frac{2}{\sigma^2 t}}}$
- Proposition 2: The difference $\mathbb{E}[M(t) - S(t)] = 1/\eta$. It roughly grows with order $\sigma t^{1/2}$.

Conclusion

By looking at the long term effect of the risk, ruin probability supplement VaR and TVaR as a informative risk measure.

Conclusion

Thank you!